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Strong Unicity for Monotone Approximation by Reciprocals of Polynomials

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This paper gives a counterexample that the strong unicity fails for best monotone approximation by reciprocals of polynomials and establishes the strong unicity of order 1/2 for some class of functions. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let $C[a, b]$ denote the space of all real continuous functions defined on an interval $[a, b]$ with the uniform norm. For a function f in $C[a, b]$, the best approximation of f by a family of monotone functions has been studied by many authors (see [1–4]). In paper [4], the approximation by monotone reciprocals of polynomials was considered. The approximation problem, which was presentrd by G. D. Taylor, is the first step for studying the approximation by monotone rationals. The characterization and unicity theorems were given in [4]. It is already known that strong unicity of the best approximation is related to the Lipschitz conditions for the best approximation operators. In this paper, we consider the strong unicity of the best monotone approximation by reciprocals of polynomials. In Section 2, at first, we give a counterexample for which the strong unicity fails; and second, in Sections 3 and 4, the unicity for some class of functions and strong unicity of order 1/2 are established, respectively.

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Let π_n denote the set of real algebraic polynomials of degree n or less. Define

$$R_n = \{r = d/p \colon p \in \pi_n,\ p(x) > 0 \text{ for all } x \text{ in } [a, b],\ d = \pm 1 \text{ or } 0\}$$
$$R_n^* = \{r \in R \colon r'(x) \geqslant 0 \text{ for all } x \text{ in } [a, b]\}.$$

For $f \in C[a, b]$, if $r_f \in R_n^*$ satisfies

$$\|f - r_f\| = \inf\{\|f - r\| : r \in R_n^*\}$$

then r_f is called a best monotone approximation of f by reciprocals of polynomials. From [4], for any $f \in C[a, b]$, the best approximation to f exists.

Remark. (1) If $n = 0$ or $f \in R_n^*$, the best approximation to f is strongly unique

(2) $f \in C[a, b]$ satisfies $\max\{f(x) \colon x \in [a, b]\} = -\min\{f(x) \colon x \in [a, b]\}$ if and only if the best approximation of f in R_n^* is $r_f = 0$ (see [4] or observe this directly by the characterization of [6]). In this case, $r_f = 0$ is not strongly unique in general. For example, let

$$f_n(x) = \begin{cases} -1/4, & x = 0 \\ -[(n+6)/(n+2)]\,x - 1/4, & 0 < x < 1/2 \\ -3/4 - 2/(n+2), & x = 1/2 \\ \dfrac{3n^2 + 11n + 6}{(n+1)(n+2)}\left(x - \dfrac{1}{2}\right) - \dfrac{3}{4} - \dfrac{2}{n+2} & 1/2 < x < 1 \\ 3/4 - 1/(n+1), & x = 1 \end{cases}$$

$$f_0(x) = \begin{cases} -1/4, & x = 0 \\ -x - 1/4, & 0 < x < 1/2 \\ -3/4, & x = 1/2 \\ 3(x - 1/2) + 3/4, & 1/2 < x < 1 \\ 3/4, & x = 1 \end{cases}$$

$$r_n(x) = -1/(nx + 1),\ x \in [0, 1]; \qquad r_0 = 0,\ x \in [0, 1].$$

Then for any $n = 1, 2, \ldots, r_n$ is a best approximation to f_n in R_n. Since $r_n \in R_n^*$, r_n is also a best approximation to f_n in R_n^*. On the other hand, r_0 is also a best approximation to f in R_n^*. Since $f_n \to f_0$ but $r_n \not\to r_0$, r_0 is not strongly unique.

By the above remarks, without loss of generality, we can assume that $n \geqslant 1, f \in C[a, b] \backslash R_n^*$, and $\max(f(x) \colon x \in [a, b]\} \neq -\min\{f(x) \colon x \in [a, b]\}$ in the following.

2. Characterization and Counterexample

The following is given in [4]:

THEOREM A. *$f \in C[a, b] \backslash R_n^*$, $r_f = d/p_f$ ($d = 1$ or -1) is a best approximation to f in R_n^* if and only if there is no element $q \in \pi_n$ such that*

$$\operatorname{sign} q(x) = \operatorname{sign}[f(x) - r_f(x)], \qquad x \in A(f, r_f)$$
$$q'(x) > 0, \qquad x \in B(r_f),$$

where

$$A(f, r_f) = \{x \in [a, b] : \|f - r_f\| = |f(x) - r_f(x)|\}$$
$$B(r_f) = \{x \in [a, b] : r'(x) = 0\}.$$

For convenience, we define the notations

$$S_1 = \{\sigma(x)(1, x, ..., x^n) : x \in A(f, r_f)\}$$
$$S_2 = \{(0, 1, 2x, ..., nx^{n-1}) : x \in B(r_f)\},$$

where $\sigma(x) = \operatorname{sign}[f(x) - r_f(x)]$.

THEOREM 2.1. *$f \in C[a, b] \backslash R_n^*$, $\max\{f(x) : x \in [a, b]\} \neq -\min\{f(x) : x \in [a, b]\}$, $r_f \in R_n^*$, and the following are equivalent to each other:*

(1) *r_f is a best approximation to f in R_n^*.*

(2) $0 \in \operatorname{co}(S_1 \cup S_2)$.

(c) *There are $x_1, ..., x_m \in A(f, r_f)$, $y_1, ..., y_k \in B(r_f)$, and $\alpha_i > 0$, $\beta_j > 0$ ($i = 1, 2, ..., m$; $j = 1, 2, ..., k$) such that for any $p \in \pi_n$*

$$\sum_{i=1}^{m} \alpha_i \sigma(x_i)\, p(x_i) + \sum_{j=1}^{k} \beta_j p'(y_j) = 0 \tag{1}$$

and in addition, $m + 2k - e \geq n + 2$, $m \geq 1$, where e is the number of points in $\{a, b\} \cap \{y_1, ..., y_k\}$.

Proof. By the "Linear Inequality Theorem" of [6] and Theorem A, the equivalence of (1) and (2) is obtained easily.

As to the equivalence of (2) and (3), what we need to do is to prove that $m \geq 1$ and $m + 2k - e \geq n + 2$. Clearly, $m \geq 1$ is true. In fact, if otherwise we would have that for any $f' \in C[a, b] \backslash R_n^*$, r_f was also a best approximation to f', which, obviously, is a contradiction. Now let's prove that $m + 2k - e \geq n + 2$. Set

$$H = (a, b) \cap [\{y_1, ..., y_k\} \backslash \{x_1, ..., x_m\}]$$

and let s be the number of points in H. Without loss of generality, assume that $H=\{y_1, ..., y_s\}$. If we can prove that $m+k+s \geqslant n+2$ then

$$m+2k-e \geqslant m+k+s \geqslant n+2$$

and thus we complete the proof.

Suppose that $K=m+k+s \leqslant n+1$, then (1) implies

$$\sum_{i=1}^{m} \alpha_i \sigma(x_i)\, x_i^t + \sum_{v=1}^{s} \gamma_v y_v^t + \sum_{j=1}^{k} \beta_j t y_j^{t-1} = 0, \qquad t=0, 1, ..., K, \tag{2}$$

where $\gamma_v = 0$, for $v=1, 2, ..., s$. Let C denote the coefficient matrix of (2) in the view of $\alpha_i \sigma(x_i)$, γ_v, β_j being unknown numbers for $i=1, 2, ..., m$, $v=1, 2, ..., s$, $j=1, 2, ..., k$, then C is the matrix of the following Birkhoff interpolation problem. Find $q \in \pi_k$ satisfying

$$\begin{aligned} q(x_i) &= a_i, & i&=1, 2, ..., m \\ q(y_v) &= b_v, & v&=1, 2, ..., s \\ q'(y_j) &= c_j, & j&=1, 2, ..., k. \end{aligned} \tag{3}$$

Since, in interpolation problem (3), the points where the interpolation condition on g' can occur without a corresponding condition on q are at most the points a and b, the incidence matrix of the above Birkhoff interpolation problem satisfies the Polya condition. Moreover, the incidence matrix has no support sequence, hence the incidence matrix is order normal by [7]. This implies that (3) has a unique solution in π_k. So (2) has only the zero solution, which is a contradiction.

EXAMPLE 2.1 (Best Approximation Which Is Not Strongly Unique). Define

$$\begin{aligned} p_f(x) &= -(x-1/\sqrt{3})^3 + c, & x &\in [-1, 1] \\ f(x) &= 1/2 - x^2 + [p_f(x)]^{-1}, & x &\in [-1, 1], \end{aligned}$$

where c is a constant with

$$\min_{[-1,1]} p_f(x) = \sqrt{2}.$$

Then $r_f = 1/p \in R_3^*$ is a best approximation to f.

In fact, $\|f - r_f\| = 1/2$ and

$$A(f, r_f) = \{-1, 0, 1\}; \qquad B(r_f) = \{1/\sqrt{3}\}$$

$$\sigma(-1) = -1, \qquad \sigma(0) = 1, \qquad \sigma(1) = -1.$$

It is easy to check that

$$(2-\sqrt{3})\,\sigma(-1)\,p(-1)+4\sigma(0)\,p(0)$$
$$+(2+\sqrt{3})\,\sigma(1)\,p(1)+2\sqrt{3}\,p'(1/\sqrt{3})=0$$

for $p(x)=1,\ x,\ x^2,\ x^3$. By Theorem 2.1, r_f is a best approximation to f in R_3^*.

For sufficiently small $\alpha>0$, set

$$p_\alpha(x)=p_f(x)-\alpha x[x^2-(1-\alpha)],\qquad x\in[-1,1].$$

Then $p'_\alpha\leqslant 0$ for all x in $[-1,1]$ and $r_\alpha=1/p_\alpha\in R_3^*$. Since $p_\alpha\to p_f$ when $\alpha\to 0$, there is a $M>0$ such that

$$1\leqslant p_\alpha(x)\,p_f(x)\leqslant M$$

for any $x\in[-1,1]$ and sufficiently small $\alpha>0$.

Claim I. For sufficiently small $\alpha>0$,

$$\|r_\alpha-r_f\|\geqslant 2\alpha(1-\alpha)^{3/2}/M3^{3/2}.$$

In fact, by [2, Sect. 2, Claim II]

$$\|p_\alpha-p_f\|=2/3^{3/2}[\alpha(1-\alpha)^{3/2}]$$

which implies that

$$\|r_\alpha-r_f\|\geqslant\frac{1}{M}\,\|p_\alpha-p_f\|=\frac{2}{M\sqrt{3^3}}\,\alpha(1-\alpha)^{3/2}.$$

Claim II. For sufficiently small $\alpha>0$,

$$\|f-r_\alpha\|\leqslant\tfrac{1}{2}+\alpha^2.$$

Proof. The following was proved in [2]:

$$\|\tfrac{1}{2}-x^2\{-\alpha x[x^2-(1-\alpha)]\}\|=\tfrac{1}{2}+\alpha^2. \tag{4}$$

For any $x\in[-1,1]$

$$|f(x)-r_\alpha(x)|=|\tfrac{1}{2}-x^2+\{-\alpha x[x^2-(1-\alpha)]/p_f(x)\,p_\alpha(x)\}|.$$

If $1/2-x^2$ and $-\alpha x[x^2-(1-\alpha)]$ have the same sign, then

$$\begin{aligned}|f(x)-r_\alpha(x)| &= |\tfrac{1}{2}-x^2|+|-\alpha x[x^2-(1-\alpha)]/p_f(x)\,p_\alpha(x)|\\ &\leqslant |\tfrac{1}{2}-x^2|+|-\alpha x[x^2-(1-\alpha)]|\\ &= |\tfrac{1}{2}-x^2-\alpha x[x^2-(1-\alpha)]|\\ &\leqslant \tfrac{1}{2}+\alpha^2.\end{aligned}$$

If $1/2-x^2$ and $-\alpha x[x^2-(1-\alpha)]$ have different signs, since

$$|\tfrac{1}{2}-x^2| \geqslant |-\alpha x[x^2-(1-\alpha)]/p_\alpha(x)\,p_f(x)|$$

$$\begin{aligned}|f(x)-r_\alpha(x)| &= |\tfrac{1}{2}-x^2|-|-\alpha x[x^2-(1-\alpha)]/p_f(x)\,p_\alpha(x)|\\ &\leqslant |\tfrac{1}{2}-x^2| \leqslant \tfrac{1}{2}+\alpha^2\end{aligned}$$

holds for sufficiently small $\alpha>0$, the proof of Claim II is complete.

Now from Claim I and Claim II

$$[\|f-r_\alpha\|-\|f-r_f\|]/\|r_\alpha-r_f\| \leqslant 3^{3/2}M\alpha/[2(1-\alpha)^{3/2}] \to 0$$

when $\alpha \to 0$. Hence r_f is not strongly unique.

3. Strong Unicity for Some Classes

LEMMA 3.1. *If $r_f \in R_n^*$ is a best approximation to f, and $q \in \pi_n$ satisfies*

(1) $q(x)\,\sigma(x) \geqslant 0$ *for any* $x \in A(f, r_f)$,

(2) $q'(x) \geqslant 0$ *for any* $x \in B(r_f)$,

(3) $x \in (a, b) \cap B(r_f)$ *and* $q'(x)=0$ *implies* $q''(x)=0$;

or if it satisfies (1), (2), *and*

(3′) $1 \leqslant \partial p_f \leqslant 2$

then $q \equiv 0$, *where* ∂p_f *means the degree of* p_f.

Proof. By the conditions (1) and (2) and Theorem 2.1, we have that $q(x_i)=0$ and $q'(y_j)=0$ for $i=1, 2, ..., m$; $j=1, 2, ..., k$.

In the case (3), we count the zeroes of $q'(x)$. Since $q''(y_j)=0$ for $y_j \in (a, b)$, $q'(x)$ has at least two zeroes at y_j for $y_j \in (a, b)$ and $q'(x)$ has at least $2k-e$ zeroes at $y_1, ..., y_k$. As to $x_1, ..., x_m$, without loss of generality, we assume $x_1 < x_2 ... < x_m$ and $q \not\equiv 0$. Since $q(x_i)=q(x_{i+1})=0$, $q'(x)$ has at least one zero z in (x_i, x_{i+1}), which is different from any y_j ($j=1, 2, ..., k$), or there is y_j such that $q'(x)$ has at least three zeroes at y_j. In fact, if otherwise, $q'(x)$ has two zeroes at any y_j, and $q'(x) \neq 0$ for any x in

(x_i, x_{i+1}) and $x \neq y_j$ $(j=1, 2, ..., k)$. Then $q'(x) \geqslant 0$ on (x_i, x_{i+1}) by Taylor expansion of $q'(x)$. This implies $q(x) \equiv 0$, which is a contradiction. Hence $q'(x)$ has at least $m-1+2k-e$ zeroes in $[a, b]$. This implies that $q'(x) \equiv 0$ since $m-1+2k-e \geqslant n+1$, and so $q(x) \equiv 0$.

In the case (3'), $p_f'(x)$ has at most one simple zero, y. Since $p_f(x)$ is a monotone function, $y=a$ or b and $k=e$, which implies that $q(x)$ has $m \geqslant n+2-2k+e \geqslant n+1$ zeroes. So $q(x) \equiv 0$ and the proof is complete.

LEMMA 3.2. *If the best approximation to f in R_n^* is $r_f = d/p_f$ ($d=1$ or -1), $r_k = d_k/p_k \in R_n^*$ ($d_k = 1$ or -1) such that*

$$\|f - r_k\| \to \|f - r_f\|, \qquad k \to \infty$$

then there is a sequence $\{r_k\}$ (denoted by itself) such that

$$r_k = d/p_k \qquad \text{and} \qquad p_k \to p_f,\ k \to \infty.$$

Proof. Without loss of generality, we assume that $d=1$. Since $\|f-r_k\| \to \|f-r_f\|$ when $k \to \infty$, then $d_k = 1$ for sufficiently large k. In fact, if otherwise, there is a $x_0 \in [a, b]$ with $f(x_0) = \|f\|$. So

$$\|f-(-1)/p_k\| \geqslant f(x_0) + 1/p_k(x_0) > f(x_0) = \|f\|$$

and

$$\|f-r_k\| - \|f-r_f\| > \|f\| - \|f-r_f\| > 0$$

which contradicts $\|f-r_k\| \to \|f-r_f\|$ when $k \to \infty$.

Since $\|f-r_k\| \to \|f-r_f\|$, $\{\|r_k\|\}$ is bounded, then $\delta = \inf\{p_k(x)$: $k=1, 2, 3, ...\} > 0$. Since $\{p_k/\|p_k\|\}$ and $\{1/\|p_k\|\}$ are bounded, we may assume that $p_k/\|p_k\| \to q(x)$ and $1/\|p_k\| \to a$ for some $q(x) \in \pi_n$ and a. Then for any $x \in A(f, r_f)$,

$$\begin{aligned}(q-ap_f)(x) &= \lim(p_k/\|p_k\| - p_f/\|p_k\|)(x) \\ &\leqslant \lim(p_k p_f)(x)/\|p_k\| \cdot [\|f-r_k\| - \|f-r_f\|] \\ &= 0.\end{aligned}$$

On the other hand, since $q'(x) \leqslant 0$ on $[a, b]$ by $p_k'(x) \leqslant 0$ on $[a, b]$, then

$$(q-ap_f)'((x) \leqslant 0, \qquad x \in B(r_f)$$

and if $x \in B(r_f) \cap (a, b)$ and $(q-ap_f)'(x) = 0$, then $q'(x) = 0$ and $q''(x) = 0$. Then

$$(q-ap_f)'' = 0$$

by $p_f''(x)=0$. Using Lemma 3.1, we have that

$$q - ap_f = 0$$

and

$$p_f = q/a = \lim_k p_k$$

since $\|q\| = 1$ and $a \neq 0$. The proof is complete.

THEOREM 3.1. *Assume $f \in C[a, b]$, $r_f = d/p_f \in R_n^*$ ($d = 1$ or -1) is a best approximation to f in R_n^*. If $\partial p_f \leqslant 2$, then r_f is strongly unique; that is, there exists $\gamma > 0$ such that*

$$\|f - r\| \geqslant \|f - r_f\| + \gamma \|r - r_f\|$$

for any $r \in R_n^*$.

Proof. Without loss of generality, we prove the theorem only for the case $d = 1$. Suppose there was a sequence $\{r_k\}$ in R_n^*, $r_k = d_k/p_k$ such that

$$R(r_k) = [\|f - r_k\| - \|f - r_f\|]/\|r_k - r_f\| \to 0$$

when $k \to \infty$. Since $R(r_k) \geqslant 1 - 2\|f - r_f\|/\|r - r_f\|$, $\{\|r_k - r_f\|\}$ is bounded, this concludes that $\|f - r_k\| \to \|f - r_f\|$. By Lemma 3.2, there is a subsequence $\{r_k\}$, denoted by itself again, such that $r_k = 1/p_k$ and $p_k \to p_f$ when $k \to \infty$. Let

$$C = \inf\{ \max_{x \in A(f, r_f)} [-\sigma(x)(p_f - h)(x)/\|p_f - h\|]:$$

$$h \in \pi_n, \|p_f - h\| \neq 0, h'(x) \leqslant 0 \text{ for all } x \in [a, b]\},$$

then $C > 0$. In fact, if $C \leqslant 0$, there is $h_m \in \pi_n$ with $h_m' \leqslant 0$ for $m = 1, 2, \ldots$ such that

$$\lim_m \max\{-\sigma(x)(p_f - h_m)(x)/\|p_f - h_m\| : x \in A(f, r_f)\} \leqslant 0.$$

Then there exists a subsequence of $\{(p_f - h_m)/\|p_f - h_m\|\}$, denoted by itself, and $q \in \pi_n$ such that

$$(p_f - h_m)/\|p_f - h_m\| \to q \qquad \text{when} \quad k \to \infty.$$

Hence

$$\sigma(x)\, q(x) \geqslant 0, \qquad x \in A(f, r_f)$$

$$q'(x) \geqslant 0, \qquad x \in B(r_f)$$

if $\partial p_f \geqslant 1$, $q(x)$ satisfies the conditions (1), (2), and (3′) in Lemma 3.1, and $q(x) \equiv 0$. If $\partial p_f = 0$, p_f is a constant, and $q(x)$ is monotone, then $q''(x) = 0$ for any $x \in B(r_f) \cap (a, b)$ and $q'(x) = 0$. Again by Lemma 3.1, $q(x) \equiv 0$, which contradicts $\|q\| = 1$. So $C > 0$. But on the other hand, for any k there exists $x \in A(f, r_f)$ such that

$$\begin{aligned} R(r_k) &\geqslant \sigma(x)(p_k - p_f)(x)/[\|r_k - r_f\|\ p_k(x)\ p_f(x)] \\ &\geqslant C\|p_k - p_f\|/[\|r_k - r_f\|\ p_k(x)\ p_f(x)] \\ &\geqslant C/[\|1/p_k p_f\|\ p_k(x)\ p_f(x)]. \end{aligned}$$

Let

$$M = \|p_f\| + 1, \qquad \delta = \min_{[a,b]} p_f(x) > 0.$$

Then for sufficiently large k,

$$\min_{[a,b]} p_k(x) \geqslant \frac{\delta}{2}, \qquad \|p_k\| \leqslant M, \qquad \text{and} \qquad R(r_f) \geqslant \frac{c\delta^2}{2M^2} > 0$$

which contradicts that $R(r_k) \to 0$ when $k \to \infty$. The proof is complete.

4. Strong Unicity of Order 1/2

Suppose $r_f = d/p_f \in R_n^*$ ($d = 1$ or -1) is a best approximation to f in R_n^*. For any $g \in \pi_n$, define

$$\|g\|' = \max\{|g(x)|, |g'(y)| : x \in A_m, y \in B_1\},$$

where $A_m = \{x_1, x_2, ..., x_m\} \subset A(f, r_f)$, $B_1 = \{y_1, y_2, ..., y_1\} \subset B(r_f)$ such that A_m and B_1 satisfy (1). $\|\cdot\|'$ is a seminorm in π_n and by Markov's inequality [8] we can easily get that, for any g in π_n, $\|g\|' \leqslant M\|g\|$ for some constant $M > 0$.

Theorem 4.1. *Suppose* $f \in C[a, b]$ *with* $\max\{f(x) : x \in [a, b]\} \neq -\min\{f(x) : x \in [a, b]\}$, r_f *is a best approximation to* f *in* R_n^*. *Then* r_f *is strongly unique of order* 1/2, *that is, for any* $N > 0$, *there is* $\gamma > 0$ *such that*

$$\|f - r\| \geqslant \|f - r_f\| + \gamma\|r_f - r\|^2$$

for any r *in* R_n^* *with* $\|r - r_f\| \leqslant N$.

Proof. When $n = 0$ or $f \in R_n^*$, the result is true. Now we assume that $n > 0$ and $f \notin R_n^*$. In this case, the best approximation $r_f = d/p_f$ ($d = -1$ or 1).

Again, we prove this only for $d=1$, since the case $d=-1$ is similar. For any $N>0$, let

$$\pi_n(f, N) = \{p \in \pi_n : p(x) \leqslant N \text{ for any } x \in [a, b]\}$$

LEMMA 4.1. *For any $N>0$, there is $\rho>0$ such that*

$$\|f-r\| \geqslant \|f-r_f\| + \rho \|p_f - p\|'$$

for any $r = d/p \in R_n^*$ ($d=1$ or -1) with $p \in \pi_n(f, N)$.

Proof. Suppose there is $r_k = d_k/p_k \in R_n^*$ ($d=1$ or -1) and $p_k \in \pi_n(f, N)$ such that when $k \to \infty$

$$R'(r_k) = [\|f-r_k\| - \|f-r_f\|]/\|p_k - p_f\|' \to 0,$$

then $\{\|p_k - p_f\|'\}$ is bounded by Markov's inequality. So $\|f-r_k\| \to \|f-r_f\|$. By Lemma 3.2, there is a subsequence $\{r_k\}$ (denoted by itself), such that $p_k \to p_f$. Let

$$C = \inf\left\{ \max_{x \in A(f, r_f)} \frac{-\sigma(x)(p_f - h)(x)}{\|p_f - h\|'} : \|p_f - h\|' \neq 0, \right.$$
$$\left. h'(x) \leqslant 0 \text{ for all } x \in [a, b] \text{ and } h \in \pi_n \right\},$$

and with the same techniques as in the proof of Theorem 3.1 we get $C>0$. Thus, for any $k = 1, 2, \ldots$, there is $x \in A(f, r_f)$ such that

$$R'(r_k) \geqslant \frac{\sigma(x)(p_k - p_f)(x)}{\|p_k - p_f\|' \, (p_k p_f)(x)} \geqslant \frac{C}{(p_k p_f)(x)},$$

which contradicts $R'(r_k) \to 0$ when $k \to \infty$. The proof of Lemma 4.1 is complete.

Now let's go back to the proof of Theorem 4.1. We prove that

$$R(r) = \frac{\|f-r\| - \|f-r_f\|}{\|r-r_f\|^2}$$

has positive lower bound in $\{r \in R_n^* : \|r - r_f\| \leqslant N\}$. If not, there exists a sequence $\{r_k\} \in R_n^*$ with $\|r_k - r_f\| \leqslant N$ such that $R(r_k) \to 0$. Then again by Lemma 3.2, there is a subsequence of $\{r_k\}$, denoted by itself again, such that $r_k = 1/p_k$ and $p_k \to p_f$ when $k \to \infty$. Appealing to the proof in [3, Theorem 3.4], we obtain that $\|p_k - p_f\|^2 \leqslant M_1 \|p_k - p_f\|'$ for some constant M_1. Since $p_k \to p_f$, $r_k \in R_n^*$, $p_k \in \pi_n(f, N_1)$ for some N_1, then by Lemma 4.1 there exists a $\rho > 0$ such that

$$\|f - r_k\| \geqslant \|f - r_f\| + \rho \|p_k - p_f\|'$$

for any $k = 1, 2, \ldots$. So

$$\begin{aligned} R(r_k) &\geqslant \rho \|p_k - p_f\|' / \|r_k - r_f\|^2 \\ &\geqslant \frac{\rho}{M_1} \|p_k - p_f\|^2 / \|r_k - r_f\|^2 \\ &\geqslant \frac{\rho\delta^2}{2M_1} \end{aligned}$$

when $k \to \infty$, which contradicts $R(r_k) \to 0$ and proves the theorem.

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